

The Construction of Dirac Wave Packets for a Fermionic Particle Non-Minimally Coupling with an External Magnetic Field

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We shall proceed with the construction of normalizable Dirac wave packets for *fermionic* particles (neutrinos) with dynamics governed by a “modified” Dirac equation with a non-minimal coupling with an external magnetic field. We are not only interested on the analytic solutions of the “modified” Dirac wave equation but also on the construction of Dirac wave packets which can be used for describing the dynamics of some observable physical quantities which are relevant in the context of the quantum oscillation phenomena. To conclude, we discuss qualitatively the applicability of this formal construction in the treatment of chiral (and flavor) oscillations in the theoretical context of neutrino physics.

KEY WORDS: dirac equation; wave packets; non-minimal coupling.

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Since those early days when Dirac had derived the relativistic wave equation for a free propagating electron (Dirac, 1928), several efforts have been produced in the literature to solve the Dirac equation with other analytical forms of interacting potentials, from central potential solutions (Esposito and Santorelli, 1999; Alhaidari, 2005) to recent theoretical attempts to describe quark confinement (Eichten and Feinberg, 1981; Baker *et al.*, 1995). In fact, obtaining exact solutions of a generic class of Dirac wave equations becomes important since, for many times, the conceptual understanding of physics can only be brought about by such solutions. These solutions also correspond to valuable means for checking and improving models and numerical methods for solving complicated physical problems. In the context in which we intend to explore the Dirac formalism, we can quote the effect of neutrino spin flipping attributed to some dynamic external (D’Olivo *et al.*, 1990) interacting processes which come from the non-minimal coupling of a magnetic moment with an external electromagnetic field (Voloshin

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et al., 1986) and which was formerly supposed to be a relevant effect in the context of the solar-neutrino puzzle by suggesting an explanation for the LSND anomaly (Athanasopoulos *et al.*, 1998; Aguilar *et al.*, 2001; Bandyopadhyay *et al.*, 2004). To be more specific, our aim in this manuscript is to try to accommodate the Dirac wave packet formalism (Itzykinson and Zuber, 1980; Bernardini and De Leo, 2004) and a further class of static characteristics of neutrinos, namely, the (electro)magnetic moment which appears in a Lagrangian with non-minimal coupling. By following this line of reasoning, we are not only interested in solving a “modified” Dirac wave equation but also in constructing Dirac wave packets which can be used for describing the dynamics of some observable physical quantities which are relevant in the context of the quantum oscillation phenomena.

Despite their electric charge neutrality, neutrinos can interact with a photon through loop (radiative) diagrams (Itzykinson and Zuber, 1980; Kim and Pevsner, 1993). The Lagrangian for the interaction between a fermionic field $\Psi(x)$ and the electromagnetic field $F^{\mu\nu}(x)$ is given by

$$\mathcal{L} = \frac{1}{2}\mu\bar{\Psi}(x)\sigma_{\mu\nu}F^{\mu\nu}(x)\Psi(x) + h.c., \quad (1)$$

with $x = (t, \mathbf{x})$, $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$ where we have not discriminated the flavor/mass mixing elements since we are initially interested just in solving the equation of the motion described by

$$\begin{aligned} i\frac{d}{dt}\Psi(x) &= \left\{ \boldsymbol{\alpha} \cdot \mathbf{p} + \beta \left[m - \left(\frac{1}{2}\mu\sigma_{\mu\nu}F^{\mu\nu}(s) + h.c. \right) \right] \right\} \Psi(x) \\ &= \{ \boldsymbol{\alpha} \cdot \mathbf{p} + \beta[m - \text{Re}(\mu)\boldsymbol{\Sigma} \cdot \mathbf{B}(x) + \text{Im}(\mu)\boldsymbol{\alpha} \cdot \mathbf{E}(x)] \} \Psi(x), \quad (2) \end{aligned}$$

where $\boldsymbol{\alpha} = \sum_{k=1}^3 \alpha_k \hat{k} = \sum_{k=1}^3 \gamma_0 \gamma_k \hat{k}$, $\beta = \gamma_0$, $\mathbf{B}(x)$ and $\mathbf{E}(x)$ are respectively the magnetic and electric fields. The real (imaginary) part of μ represents the magnetic (electric) dipole moment of the mass eigenstate represented by $\Psi(x)$. But it can be demonstrated (Kim and Pevsner, 1993) that, for Dirac neutrinos, the electric moment must vanish unless CP is violated, and, for Majorana neutrinos, the electric moment vanishes if CPT invariance is assumed. In the standard $SU(2)_L \otimes U(1)_Y$ electroweak theory (Glashow, 1961; Weinberg, 1967; Salam, 1968), if a positive chirality neutrino is an $SU(2)_L$ singlet, the expression for μ can be found from Feynman diagrams for magnetic momentum corrections (Kim and Pevsner, 1993) and turns out to be proportional to the neutrino mass (matrix),

$$\mu = \frac{3eG}{8\sqrt{2}\pi^2} m = \frac{3m_e G}{4\sqrt{2}\pi^2} \mu_B m_\nu = 2.7 \times 10^{-10} \mu_B \frac{m_\nu}{m_N} \quad (3)$$

where G is the Fermi constant and m_N is the nucleon mass. In principle, for $m_\nu \approx 1$ eV, the magnetic moment introduced by this formula is exceedingly small to be detected or to affect astrophysical or physical processes.

From the theoretical point of view, the physical implications of the non-minimal coupling with an external magnetic field can then be studied by means of the eigenvalue problem represented by the Hamiltonian equation

$$H(\mathbf{p})\Psi_n(\mathbf{p}) = \{\boldsymbol{\alpha} \cdot \mathbf{p} + \beta[m - \text{Re}(\mu)\boldsymbol{\Sigma} \cdot \mathbf{B}]\}\Psi_n(\mathbf{p}), \quad (4)$$

which represents the time evolution operator of a spinor $\Psi(x)$ for times subsequent to the creation ($t = 0$). The most general solution to the above eigenvalue problem is represented by

$$E_n(\mathbf{p}) = \pm E_s(\mathbf{p}) = \pm \sqrt{m^2 + \mathbf{p}^2 + \mathbf{a}^2 + (-1)^s 2\sqrt{m^2 \mathbf{a}^2 + (\mathbf{p} \times \mathbf{a})^2}}, \quad s = 1, 2, \quad (5)$$

where $\mathbf{a} = \text{Re}(\mu)\mathbf{B}$ and the complete set of orthonormal eigenstates corresponds to

$$\begin{aligned} (\Psi^+(p_s))^\dagger &= -N(p_s) \left\{ \sqrt{\frac{A_s^-}{A_s^+}}, \sqrt{\frac{\alpha_s^+}{\alpha_s^-}}, \sqrt{\frac{A_s^- \alpha_s^+}{A_s^+ \alpha_s^-}}, -1 \right\} \\ (\Psi^-(p_s))^\dagger &= -N(p_s) \left\{ \sqrt{\frac{A_s^-}{A_s^+}}, -\sqrt{\frac{\alpha_s^-}{\alpha_s^+}}, -\sqrt{\frac{A_s^- \alpha_s^-}{A_s^+ \alpha_s^+}}, -1 \right\}, \end{aligned} \quad (6)$$

where p_s is the relativistic *quadrimentum*, $p_s = (E_s(\mathbf{p}), \mathbf{p})$, $N(p_s)$ is the normalization constant and

$$A_s^\pm = \Delta_s^2(\mathbf{p}) \pm 2m|\mathbf{a}| - \mathbf{a}^2, \quad \alpha_s^\pm = 2E_s(\mathbf{p})|\mathbf{a}| \pm (\Delta_s^2(\mathbf{p}) + \mathbf{a}^2)$$

with

$$\Delta_s^2(\mathbf{p}) = E_s^2(\mathbf{p}) - (m^2 + \mathbf{p}^2) + \mathbf{a}^2. \quad (7)$$

We can observe that the components of $\Psi_s^\pm(\mathbf{p})$ does not satisfy any auxiliary conditions, namely, at a given time t , they are independent functions of \mathbf{p} and the eigenspinors does not represent neither chirality nor helicity eigenstates. Thus, if we seek the plane wave decomposition as

$$\begin{aligned} \Psi^+(x) &= \exp[-i(E_s(\mathbf{p})t - \mathbf{p} \cdot \mathbf{x})]\mathcal{U}(p_s), \quad \text{for positive frequencies and} \\ \Psi^-(x) &= \exp[i(E_s(\mathbf{p})t - \mathbf{p} \cdot \mathbf{x})]\mathcal{V}(p_s), \quad \text{for negative frequencies} \end{aligned} \quad (8)$$

the time evolution of a plane wave packet solution $\Psi(t, x)$ in this circumstances can be written as

$$\begin{aligned} \Psi(t, \mathbf{x}) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{s=1,2} \{b(p_s)\mathcal{U}(p_s) \exp[-iE_s(\mathbf{p})t] \\ &+ d^*(\tilde{p}_s)\mathcal{V}(\tilde{p}_s) \exp[+iE_s(\mathbf{p})t]\} \exp[i\mathbf{p} \cdot \mathbf{x}], \quad \text{with } \tilde{p}_s = (E_s, -\mathbf{p}), \end{aligned} \quad (9)$$

which, however, requires some extensive mathematical manipulations for explicitly constructing the dynamics of an operator \mathcal{O} as

$$\mathcal{O}(t) = \int d^3\mathbf{x} \Psi^\dagger(t, \mathbf{x}) \mathcal{O} \Psi(t, \mathbf{x}). \tag{10}$$

Meanwhile, if the quoted observables like the chirality γ^5 , the helicity $\boldsymbol{\Sigma} \cdot \mathbf{p}$ or even the spin projection onto the magnetic field $\boldsymbol{\Sigma} \cdot \mathbf{B}$ commute with the Hamiltonian H , we could reconfigure the above solutions to simpler ones. To illustrate this point we shall limit our analysis to very restrictive spatial configurations of \mathbf{B} so that, as a first attempt, we could calculate the observable expectation values which appear in Eq. (10). Let us then assume that the magnetic field \mathbf{B} is either orthogonal or parallel to the momentum \mathbf{p} . For both of these cases the spinor eigenstates can now be decomposed into bi-spinors which are eigenstates of the spin projection operator $\boldsymbol{\sigma} \cdot \mathbf{B}$ as

$$\mathcal{U}(p_s) = N^+(p_s) \begin{bmatrix} \varphi^+(p_s) \\ \chi^+(p_s) \end{bmatrix} \tag{11}$$

and

$$\mathcal{V}(p_s) = N^-(p_s) \begin{bmatrix} \varphi^-(p_s) \\ \chi^-(p_s) \end{bmatrix}, \tag{12}$$

i. e. beside of being energy eigenstates, the general solutions $\mathcal{U}(p_s)$ and $\mathcal{V}(p_s)$ will become eigenstates of the operator $\boldsymbol{\Sigma} \cdot \mathbf{B}$ and, equivalently, of $\boldsymbol{\Sigma} \cdot \mathbf{a}$. The Eq. (4) can thus be decomposed into a pair of coupled equations like

$$\begin{aligned} (\pm E_s - m + \boldsymbol{\sigma} \cdot \mathbf{a}) \varphi_s^\pm &= \pm \boldsymbol{\sigma} \cdot \mathbf{p} \chi_s^\pm, \\ (\pm E_s + m - \boldsymbol{\sigma} \cdot \mathbf{a}) \chi_s^\pm &= \pm \boldsymbol{\sigma} \cdot \mathbf{p} \varphi_s^\pm, \end{aligned} \tag{13}$$

where we have suppressed the p_s dependence. The constraint imposed by the above equations leads to following representation for the energy eigenstates,

$$\mathcal{U}(p_s) = N^+(p_s) \begin{bmatrix} \varphi_s^+ \\ \frac{(E_s + m + \boldsymbol{\sigma} \cdot \mathbf{a}) \boldsymbol{\sigma} \cdot \mathbf{p}}{[(E_s + m)^2 + \mathbf{a}^2]} \varphi_s^+ \end{bmatrix} \tag{14}$$

and

$$\mathcal{V}(p_s) = N^-(p_s) \begin{bmatrix} \frac{(E_s + m + \boldsymbol{\sigma} \cdot \mathbf{a}) \boldsymbol{\sigma} \cdot \mathbf{p}}{[(E_s + m)^2 + \mathbf{a}^2]} \chi_s^- \\ \chi_s^- \end{bmatrix}. \tag{15}$$

Since $\varphi_{1,2}^+ \equiv \chi_{1,2}^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represent orthonormal eigenvectors of $\boldsymbol{\sigma} \cdot \mathbf{a}$, we can immediately deduce the orthogonality relations

$$\begin{aligned} \mathcal{U}^\dagger(p_s)\mathcal{U}(p_r) &= \mathcal{V}^\dagger(p_s)\mathcal{V}(p_r) = \delta_{sr}, \\ \mathcal{U}^\dagger(p_s)\gamma_0\mathcal{V}(p_r) &= \mathcal{V}^\dagger(p_s)\gamma_0\mathcal{U}(p_r) = 0. \end{aligned} \quad (16)$$

The above results can be obtained by simply introducing the commuting relation $[\boldsymbol{\sigma} \cdot \mathbf{p}, \boldsymbol{\sigma} \cdot \mathbf{B}] = 0$ which is derived when $\mathbf{p} \times \mathbf{B} = 0$ and the anti-commuting relation $\{\boldsymbol{\sigma} \cdot \mathbf{p}, \boldsymbol{\sigma} \cdot \mathbf{B}\}$ when $\mathbf{p} \cdot \mathbf{B} = 0$. Now let us separately summarize other relevant information for both of these restrictive cases. If $\mathbf{p} \cdot \mathbf{B} = 0$, the eigenspinor representation can be reduced to

$$\mathcal{U}(p_s) = \sqrt{\frac{\epsilon_0 + m}{2\epsilon_0}} \begin{bmatrix} \varphi_s^+ \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{\epsilon_0 + m} \varphi_s^+ \end{bmatrix} \quad (17)$$

and

$$\mathcal{V}(p_s) = \sqrt{\frac{\epsilon_0 + m}{2\epsilon_0}} \begin{bmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{\epsilon_0 + m} \chi_s^- \\ \chi_s^- \end{bmatrix}, \quad (18)$$

with $\epsilon_0 = \sqrt{\mathbf{p}^2 + m^2}$ and the energy eigenvalues

$$\pm E_s = \pm[\epsilon_0 + (-1)^s |\mathbf{a}|]. \quad (19)$$

The closure relations can be constructed in terms of

$$\begin{aligned} \sum_{s=1,2} \mathcal{U}(p_s) \otimes \mathcal{U}^\dagger(p_s) \gamma_0 &= \frac{\gamma_\mu p_0^\mu + m}{2\epsilon_0} \sum_{s=1,2} \left[\frac{1 - (-1)^s \gamma_0 \boldsymbol{\Sigma} \cdot \hat{\mathbf{a}}}{2} \right] \\ - \sum_{s=1,2} \mathcal{V}(p_s) \otimes \mathcal{V}^\dagger(p_s) \gamma_0 &= \frac{-\gamma_\mu p_0^\mu + m}{2\epsilon_0} \sum_{s=1,2} \left[\frac{1 - (-1)^s \gamma_0 \boldsymbol{\Sigma} \cdot \hat{\mathbf{a}}}{2} \right], \end{aligned} \quad (20)$$

where $p_0 = (\epsilon_0, \mathbf{p})$. If $\mathbf{p} \times \mathbf{B} = 0$, the eigenspinor representation can be reduced to

$$\mathcal{U}(p_s) = \sqrt{\frac{E_s + m_s}{2E_s}} \begin{bmatrix} \varphi_s^+ \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_s + m_s} \varphi_s^+ \end{bmatrix} \quad (21)$$

and

$$\mathcal{V}(p_s) = \sqrt{\frac{E_s + m_s}{2E_s}} \begin{bmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_s + m_s} \chi_s^- \\ \chi_s^- \end{bmatrix}, \quad (22)$$

with $m_s = m + (-1)^s |\mathbf{a}|$ and the energy eigenvalues

$$\pm E_s = \pm \sqrt{\mathbf{p}^2 + m_s^2}. \tag{23}$$

In this case, the closure relations can be constructed in terms of

$$\begin{aligned} \sum_{s=1,2} \mathcal{U}(p_s) \otimes \mathcal{U}^\dagger(p_s) \gamma_0 &= \sum_{s=1,2} \left\{ \frac{\gamma_\mu p_s^\mu + m_s}{2E_s} \left[\frac{1 - (-1)^s \boldsymbol{\Sigma} \cdot \hat{\mathbf{a}}}{2} \right] \right\} \\ - \sum_{s=1,2} \mathcal{V}(p_s) \otimes \mathcal{V}^\dagger(p_s) \gamma_0 &= \sum_{s=1,2} \left\{ \frac{-\gamma_\mu p_s^\mu + m_s}{2E_s} \left[\frac{1 - (-1)^s \boldsymbol{\Sigma} \cdot \hat{\mathbf{a}}}{2} \right] \right\}. \end{aligned} \tag{24}$$

Finally, the calculation of the expectation value of $\mathcal{O}(t)$ is substantially simplified when we substitute the above closure relations into the wave packet expression of Eq. (9). To clarify this point, we suppose the initial condition over $\Psi(0, \mathbf{x})$ is the Fourier transform of the weight function

$$\varphi(\mathbf{p} - \mathbf{p}_i) w = \sum_{s=1,2} \{ b(p_s) \mathcal{U}(p_s) + d^*(\tilde{p}_s) \mathcal{V}(\tilde{p}_s) \} \tag{25}$$

so that

$$\Psi(0, \mathbf{x}) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \varphi(\mathbf{p} - \mathbf{p}_i) \exp[i\mathbf{p} \cdot \mathbf{x}] w \tag{26}$$

where w is some fixed normalized spinor. By using the orthogonality properties of Eq. (16), we find (Itzykson and Zuber, 1980)

$$\begin{aligned} b(p_s) &= \varphi(\mathbf{p} - \mathbf{p}_i) \mathcal{U}^\dagger(p_s) w, \\ d^*(\tilde{p}_s) &= \varphi(\mathbf{p} - \mathbf{p}_i) \mathcal{V}^\dagger(\tilde{p}_s) w. \end{aligned} \tag{27}$$

These coefficients carry important physical information. For *any* initial state $\Psi(0, x)$ given by Eq. (26), the negative frequency solution coefficient $d^*(\tilde{p}_s)$ necessarily provides a non-null contribution to the time evolving wave packet. This obliges us to take the complete set of Dirac equation solutions to construct the wave packet. Only if we consider the initial spinor w being a positive energy ($E_s(\mathbf{p})$) and momentum (\mathbf{p}) eigenstate, the contribution due to the negative frequency solutions $d^*(\tilde{p}_s)$ will become null and we will have a simple expression for the time evolution of any physical observable.

As we have already noticed, under the point of view of physical applicability, the above discussion can be useful in describing the dynamics of propagating neutrinos which non-minimally couple with an approximately constant external magnetic field \mathbf{B} . In fact, the advent of the neutrino physics (Zuber, 1998; Scholberg, 2003; Alberico and Bilenky, 2004) has stirred up an increasing number of theoretical works where the quantum oscillation mechanisms (Bernardini and De Leo, 2005; Beuthe, 2003; Giunti and Kim, 1998;

Giunti, 2002; McKeown and Vogel, 2004) have been deeply analyzed. In particular, the correspondence between helicity and chirality has frequently caused some confusion in the literature where the concept of helicity has been erroneously used in the place of chirality. In the standard model of flavor-changing interactions, neutrinos with positive chirality are decoupled from neutrino absorbing charged weak currents (De Leo and Rotelli, 1998). Consequently, such positive chirality neutrinos become sterile with respect to weak interactions. By reporting to the formalism with Dirac wave packets (Itzykson and Zuber, 1980; Bernardini and De Leo, 2004) which leads to the study of chiral oscillations (De Leo and Rotelli, 1998) (in vacuum), with the formal procedures presented in this manuscript, one could obtain the *immediate* description of chiral oscillations and spin flipping in terms of the Hamiltonian (2) dynamics by recurring to the following dynamic equation

$$\frac{d}{dt}\langle\gamma^5\rangle = 2i\langle\gamma_0\gamma_5[m - \text{Re}(\mu)\boldsymbol{\Sigma} \cdot \mathbf{B}]\rangle \quad (28)$$

and

$$\frac{d}{dt}\langle h\rangle = \frac{1}{2}\text{Re}(\mu)\langle\gamma_0(\boldsymbol{\Sigma} \times \mathbf{B})\rangle \cdot \hat{\mathbf{p}}, \quad (29)$$

where the particle helicity is defined as the projection of the spin angular momentum onto the vector momentum direction $h = \frac{1}{2}\boldsymbol{\Sigma} \cdot \hat{\mathbf{p}}$. These results are obtained by means of a couple of very simple calculations, nevertheless, we believe that, in spite of its simplicity, it is important in the large context of quantum oscillation phenomena. We still remark that, in the standard treatment of vacuum neutrino oscillations, the use of scalar mass-eigenstate wave packets made up exclusively of positive frequency plane-wave solutions is usually (implicitly) assumed. Although the standard oscillation formula could give the correct result when *properly* interpreted, a satisfactory description of fermionic (spin one-half) particles requires the use of the Dirac equation as evolution equation for the mass-eigenstates. Consequently, the spinorial form and the interference between positive and negative frequency components of the mass-eigenstate wave packets lead to the possibility of chiral coupled with flavor oscillations (Bernardini and De Leo, 2005) which, eventually, deserves a very careful investigation for cases of neutrinos which propagate through external interacting (magnetic) fields.

To summarize, we would have not been honest if, by considering an arbitrary spatial configuration for the magnetic field, we had ignored the complete analysis of the general case comprised by Eqs. (4)–(6). Meanwhile, such a peculiar observation leads to the formal connection between quantum oscillation phenomena and a very different field. It concerns with the curious fact that the above complete (general) expressions for propagating wave packets do not satisfy the standard dispersion relations like $E^2 = m^2 + \mathbf{p}^2$ excepting by the two particular cases where $E_s(\mathbf{p})^2 = m_s^2 + \mathbf{p}^2$ for $\mathbf{p} \times \mathbf{B} = 0$ or $\epsilon_0^2 = m^2 + \mathbf{p}^2$ for $\mathbf{p} \cdot \mathbf{B} = 0$. By principle,

it could represent an inconvenient obstacle which forbids the extension of these restrictive cases to a general one. However, we believe that it can also represent a starting point in discussing dispersion relations which may be incorporated into frameworks encoding the breakdown (or the validity) of Lorentz invariance.

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